

# Manifestly Covariant Approach to Bargmann-Wigner Fields (II): From spin-frames to Bargmann-Wigner spinors

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## Abstract

The Bargmann-Wigner (BW) scalar product is a particular case of a larger class of scalar products parametrized by a family of world-vectors. The choice of null and  $p$ -dependent world-vectors leads to BW amplitudes which behave as local  $SU(2)$  spinors (BW-spinors) if *passive* transformations are concerned. The choice of null directions leads to a simplified formalism which allows for an application of ordinary, manifestly covariant spinor techniques in the context of infinite dimensional unitary representations of the Poincaré group.

## I. INTRODUCTION

In the first part of this work [1] (henceforth called “Part I”) we have introduced the most general form of momentum-space Bargmann-Wigner (BW) scalar products [2]. The generalized products were shown to depend on a family of world-vectors  $\{t_k^a\}$  which can be momentum-dependent.

The standard form of the product introduced by Bargmann and Wigner in [2] corresponds to an (implicit) choice of a single future-pointing, timelike, and momentum-independent vector  $t^a$ . This vector implicitly fixes a decomposition of the Minkowski space into “time” and “space” which, accordingly, is used to divide  $SL(2, C)$  transformations into “boosts” and “rotations”. For fields defined on  $m \neq 0$  mass shell the “rotations” possess finite dimensional representations which are unitary with respect to the scalar product defined by  $t^a$ ; “boosts” are represented unitarily by

momentum-dependent “rotations”, the so-called Wigner rotations [4]. The momentum dependence of Wigner rotations makes the representation infinite-dimensional.

An assumption that generators of *unitary* representations of symmetry groups are directly related to observable quantities is one of the most fundamental postulates of quantum theory. On the other hand, *manifestly covariant* formulations of classical relativistic theories proved to be incredibly efficient from both physical and mathematical point of view. The fact that the mentioned implicit decomposition into “time” and “space” is built so deeply into the structure of the unitary representation makes it practically impossible to discuss quantum theories in a manifestly covariant way directly at the level of physical states. In order to have covariant formulas in the standard approach one has to switch to spinor wave functions which do not have a direct probability interpretation. And *vice versa*: If one wants to directly deal with probability amplitudes one has to switch from spinors to BW amplitudes which are noncovariant.

The difficulties mentioned above motivated the author of this paper to look for a manifestly covariant reformulation of the standard unitary representations of the Poincaré group. In the next paper of this series we shall discuss the most general form of generators obtained if one keeps the world-vectors  $\{t_k^a\}$  arbitrary.

In this paper, however, we shall turn to another particular form of unitary representations which seems to have been overlooked until now: Those arising if one takes  $t^a$  *null* and  $p$ -*dependent*. The “null formulation” is elegant and simple. Its main technical advantage over the “timelike” one is based on the factorization property  $t^{AA'} = \tau^A \bar{\tau}^{A'}$  typical of null world-vectors. This property leads directly from spinors to BW wave functions, but the wave functions so obtained are not equivalent to helicity amplitudes. The wave functions are  $SL(2, C)$  scalars if one considers *active* spinor transformations of the spinor BW fields, but become  $SU(2)$  spinors (BW-spinors) if one considers *passive* transformations. BW-spinors can be treated by standard spinor methods. In addition to symplectic  $\varepsilon$ -spinors used to raise and lower BW-indices we introduce metric  $\varsigma$ -spinors used to express covariantly the BW scalar products.

The proofs given below are based on particularly chosen fields of spin-frames associated with 4-momenta. The explicit form of the spin-frames is given in Sec. II. In the rest of the paper we do

not make use of the explicit forms themselves but take advantage of some of their properties.

## II. SPIN-FRAMES ASSOCIATED WITH FUTURE POINTING 4-MOMENTA

Let  $S \in SL(2, C)$ . The representations  $(\frac{1}{2}, 0)$ ,  $(0, \frac{1}{2})$ ,  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ , and  $(1, 1)$  of  $S$  will be denoted by  $S_A{}^B$ ,  $\bar{S}_{A'}{}^{B'}$ ,  $S_\alpha{}^\beta$ , and  $\Lambda_a{}^b$ , respectively. We shall occasionally skip the indices if no ambiguities arise. Spinor transformations of upper- and lower-index spinors are

$$T(S)\phi_A = S_A{}^B\phi_B, \quad (1)$$

$$T(S)\phi^A = \phi^B S^{-1}{}_B{}^A = -S^A{}_B\phi^B. \quad (2)$$

Analogous transformations hold for primed spinors. The convention differs slightly from this used in [3] (cf. Eq. (3.6.1)).

### A. Explicit spin-frame for $m \neq 0$

Consider an arbitrary  $p$ -independent spinor  $\nu^A \neq 0$ . Let  $\omega^a = \omega^A \bar{\omega}^{A'}$ ,  $\pi^a = \pi^A \bar{\pi}^{A'}$ , where

$$\omega^A = \left[ \frac{m}{\sqrt{2}} \right]^{1/2} \frac{\nu^A}{\sqrt{p^{BB'} \nu_B \bar{\nu}_{B'}}} = \omega^A(\nu, p) \quad (3)$$

$$\pi^A = \left[ \frac{\sqrt{2}}{m} \right]^{1/2} \frac{p^{AA'} \bar{\nu}_{A'}}{\sqrt{p^{BB'} \nu_B \bar{\nu}_{B'}}} = \pi^A(\nu, p). \quad (4)$$

Spinors (3), (4) satisfy

$$\omega^a p_a = m/\sqrt{2}, \quad (5)$$

$$p^a = \frac{m}{\sqrt{2}} (\pi^a + \omega^a), \quad (6)$$

$$\omega^A(\nu, p) \pi_A(\mu, p) = \bar{\omega}^{A'}(\mu, p) \bar{\pi}_{A'}(\nu, p), \quad (7)$$

$$\pi^A(\nu, p) \pi_A(\mu, p) = \bar{\omega}^{A'}(\nu, p) \bar{\omega}_{A'}(\mu, p), \quad (8)$$

$$\omega_A(\nu, p) \pi^A(\nu, p) = 1, \quad (9)$$

$$S_A{}^B \omega_B(\nu, \Lambda^{-1}p) = \omega_A(S\nu, p), \quad (10)$$

$$S_A{}^B \pi_B(\nu, \Lambda^{-1}p) = \pi_A(S\nu, p), \quad (11)$$

where  $p^a$  is future-pointing and non-null, and  $\nu_A$ ,  $\mu_A$  are arbitrary.

## B. Explicit spin-frame for $m = 0$

Let  $n^a$  be  $p$ -independent, timelike and future-pointing, and let  $\nu_A \neq 0$  be also  $p$ -independent and arbitrary. Define

$$\pi^A(\nu, p) = \frac{p^{AA'} \bar{\nu}_{A'}}{\sqrt{p^{BB'} \nu_B \bar{\nu}_{B'}}}, \quad (12)$$

$$\omega^A(\nu, n, p) = \frac{n^{AA'} \bar{\pi}_{A'}(\nu, p)}{n^a p_a} = \frac{n^{AA'} p^B{}_{A'} \nu_B}{n^a p_a \sqrt{p^b \nu_b}}, \quad (13)$$

satisfying the spin-frame condition

$$\pi_A(\nu, p) \omega^A(\nu, p) = 1 \quad (14)$$

(note that the roles of  $\pi$  and  $\omega$  are reversed with respect to the massive case — this is consistent with the notation from Part I). Similarly to (10), (11) we find

$$S_A{}^B \omega_B(\nu, n, \Lambda^{-1} p) = \omega_A(S\nu, \Lambda n, p), \quad (15)$$

$$S_A{}^B \pi_B(\nu, \Lambda^{-1} p) = \pi_A(S\nu, p). \quad (16)$$

An explicit calculation shows also that

$$\pi_A(\nu, p) \bar{\pi}_{A'}(\nu, p) = \pi_A(\mu, p) \bar{\pi}_{A'}(\mu, p) = p_{AA'}, \quad (17)$$

which implies

$$\pi_A(\nu, p) = \frac{p^{BB'} \mu_B \bar{\nu}_{B'}}{|p^{CC'} \mu_C \bar{\nu}_{C'}|} \pi_A(\mu, p). \quad (18)$$

Therefore two  $\pi$ -spinors having the same flagpole  $p^a$  differ by a phase.

## III. PASSIVE TRANSFORMATIONS OF BW AMPLITUDES ( $M \neq 0$ )

Let  $S^a(p)_\alpha{}^\beta$  denote the momentum-space Pauli-Lubanski (P-L) vector for the bispinor ( $m \neq 0$ ) representation and  $S(\omega, p)_\alpha{}^\beta = \omega^a S_a(p)_\alpha{}^\beta$ .

A Fourier transform of the Dirac bispinor expanded in eigenstates of  $S(\omega, p)_\alpha{}^\beta$  is (cf. Eq. (127) in Part I)

$$\psi_{\pm}(\pm p)_{\alpha} = \begin{pmatrix} \psi_{\pm}(\pm p)_A^0 \\ \psi_{\pm}(\pm p)_{A'}^1 \end{pmatrix} = -N \begin{pmatrix} \mp \omega_A(\nu, p) f_{\pm}^{(+)}(\nu, \pm p) + \pi_A(\nu, p) f_{\pm}^{(-)}(\nu, \pm p) \\ \bar{\pi}_{A'}(\nu, p) f_{\pm}^{(+)}(\nu, \pm p) \pm \bar{\omega}_{A'}(\nu, p) f_{\pm}^{(-)}(\nu, \pm p) \end{pmatrix} \quad (19)$$

where  $N = \left[ \frac{m}{\sqrt{2}} \right]^{1/2}$ . The use of the italic font distinguishes the “BW-indices” (“0”, “1”) and the ordinary spinor ones (“0”, “1”). The amplitudes (cf. Appendix)

$$N^{-1} \bar{\omega}^{A'}(\nu, p) \psi_{\pm}(\pm p)_{A'}^1 = N^{-1} \omega^{\alpha}(\nu, p) \psi_{\pm}(\pm p)_{\alpha}^1 = f_{\pm}^{(+)}(\nu, \pm p) = f_{\pm}^1(\nu, \pm p), \quad (20)$$

$$N^{-1} \omega^A(\nu, p) \psi_{\pm}(\pm p)_A^0 = N^{-1} \omega^{\alpha}(\nu, p) \psi_{\pm}(\pm p)_{\alpha}^0 = f_{\pm}^{(-)}(\nu, \pm p) = f_{\pm}^0(\nu, \pm p), \quad (21)$$

$$N^{-1} \omega^A(\nu, p) \overline{\psi_{\pm}(\pm p)_{A'}^1} = N^{-1} \bar{\omega}^{\alpha'}(\nu, p) \bar{\psi}_{\pm}(\pm p)_{\alpha'}^0 = \overline{f_{\pm}^{(+)}(\nu, \pm p)} = \bar{f}_{\pm}^0(\nu, \pm p), \quad (22)$$

$$N^{-1} \bar{\omega}^{A'}(\nu, p) \overline{\psi_{\pm}(\pm p)_A^0} = N^{-1} \bar{\omega}^{\alpha'}(\nu, p) \bar{\psi}_{\pm}(\pm p)_{\alpha'}^1 = \overline{f_{\pm}^{(-)}(\nu, \pm p)} = \bar{f}_{\pm}^1(\nu, \pm p) \quad (23)$$

are  $SL(2, C)$  scalars. The “(±)” indices refer to “signs of spin projections in null directions”, so are not equivalent to signs of helicity (which correspond to future timelike directions). The signs “±” written without braces are signs of energy.

A general BW field is a direct sum of inequivalent spinor representations. The numerical (0 or 1) indices were introduced in Part I to distinguish between “different components” of the field (that is those belonging to inequivalent representations constituting the direct sum) and as such play a role of a binary numbering of the components. The possibility of identifying “+” with 0 and “−” with 1 is a particular property of the null formalism introduced in Part I.

Spacetime traslations are represented in momentum representation unitarily by one-dimensional phase factors. In the following sections we shall concentrate only on the nontrivial part of the unitary representation: the infinite dimensional representation of  $SL(2, C)$ .

Let  $S \in SL(2, C)$ . An active bispinor transformation of the Dirac field

$$\begin{aligned} \psi'_{\pm}(\pm p)_{\alpha} &= -N \begin{pmatrix} \mp \omega_A(\nu, p) f'_{\pm}^1(\nu, \pm p) + \pi_A(\nu, p) f'_{\pm}^0(\nu, \pm p) \\ \bar{\pi}_{A'}(\nu, p) f'_{\pm}^1(\nu, \pm p) \pm \bar{\omega}_{A'}(\nu, p) f'_{\pm}^0(\nu, \pm p) \end{pmatrix} \\ &= S_{\alpha}^{\beta} \psi_{\pm}(\pm \Lambda^{-1} p)_{\beta} \\ &= -N \begin{pmatrix} S_A^B & 0 \\ 0 & \bar{S}_{A'}^{B'} \end{pmatrix} \begin{pmatrix} \mp \omega_B(\nu, \Lambda^{-1} p) f_{\pm}^1(\nu, \pm \Lambda^{-1} p) + \pi_B(\nu, \Lambda^{-1} p) f_{\pm}^0(\nu, \pm \Lambda^{-1} p) \\ \bar{\pi}_{B'}(\nu, \Lambda^{-1} p) f_{\pm}^1(\nu, \pm \Lambda^{-1} p) \pm \bar{\omega}_{B'}(\nu, \Lambda^{-1} p) f_{\pm}^0(\nu, \pm \Lambda^{-1} p) \end{pmatrix} \end{aligned} \quad (24)$$

induces a passive transformation of the amplitudes

$$\begin{pmatrix} f'_{\pm}{}^0(\nu, \pm p) \\ f'_{\pm}{}^1(\nu, \pm p) \end{pmatrix} = \begin{pmatrix} -\omega^A(\nu, p) S_A{}^B \pi_B(\nu, \Lambda^{-1} p), & \pm \omega^A(\nu, p) S_A{}^B \omega_B(\nu, \Lambda^{-1} p) \\ \mp \bar{\omega}^{A'}(\nu, p) \bar{S}_{A'}{}^{B'} \bar{\omega}_{B'}(\nu, \Lambda^{-1} p), & -\bar{\omega}^{A'}(\nu, p) \bar{S}_{A'}{}^{B'} \bar{\pi}_{B'}(\nu, \Lambda^{-1} p) \end{pmatrix} \begin{pmatrix} f_{\pm}^0(\nu, \pm \Lambda^{-1} p) \\ f_{\pm}^1(\nu, \pm \Lambda^{-1} p) \end{pmatrix}. \quad (25)$$

Using (10), (11) we arrive at the following form of the transformation (25)

$$\begin{pmatrix} f'_{\pm}{}^0(\nu, \pm p) \\ f'_{\pm}{}^1(\nu, \pm p) \end{pmatrix} = \begin{pmatrix} \omega_A(\nu, p) \pi^A(S\nu, p), & \mp \omega_A(\nu, p) \omega^A(S\nu, p) \\ \pm \bar{\omega}_{A'}(\nu, p) \bar{\omega}^{A'}(S\nu, p), & \bar{\omega}_{A'}(\nu, p) \bar{\pi}^{A'}(S\nu, p) \end{pmatrix} \begin{pmatrix} f_{\pm}^0(\nu, \pm \Lambda^{-1} p) \\ f_{\pm}^1(\nu, \pm \Lambda^{-1} p) \end{pmatrix}. \quad (26)$$

Formula (26) shows that the “BW-indices” play a dual role analogous to spinor indices if one considers passive transformations of the BW amplitudes. To distinguish between the ordinary spinor indices and the BW indices we shall denote the latter by calligraphic letters. Therefore Eq. (26) can be written in a compact form as

$$f'_{\pm}{}^{\mathcal{A}}(\nu, \pm p) = U(S) f_{\pm}^{\mathcal{A}}(\nu, \pm p) = \mathcal{U}_{\pm}(S, \nu, p)^{\mathcal{A}}{}_{\mathcal{B}} f_{\pm}^{\mathcal{B}}(\nu, \pm \Lambda^{-1} p). \quad (27)$$

Complex conjugated amplitudes  $\bar{f}^0 = \overline{f^1}$ ,  $\bar{f}^1 = \overline{f^0}$  transform according to

$$\bar{f}'_{\pm}{}^{\mathcal{A}}(\nu, \pm p) = \overline{U(S)} \bar{f}_{\pm}^{\mathcal{A}}(\nu, \pm p) = \bar{\mathcal{U}}_{\pm}(S, \nu, p)^{\mathcal{A}}{}_{\mathcal{B}} \bar{f}_{\pm}^{\mathcal{B}}(\nu, \pm \Lambda^{-1} p). \quad (28)$$

#### IV. UNITARITY AND UNIMODULARITY OF BW TRANSFORMATION MATRICES

The matrices  $\mathcal{U}^{\mathcal{A}}{}_{\mathcal{B}} = \mathcal{U}_{\pm}(S, \nu, p)^{\mathcal{A}}{}_{\mathcal{B}}$  are unimodular

$$\varepsilon_{\mathcal{A}\mathcal{B}} = \mathcal{U}^{\mathcal{C}}{}_{\mathcal{A}} \mathcal{U}^{\mathcal{D}}{}_{\mathcal{B}} \varepsilon_{\mathcal{C}\mathcal{D}} \quad (29)$$

$$\varepsilon^{\mathcal{A}\mathcal{B}} = \mathcal{U}^{\mathcal{A}}{}_{\mathcal{C}} \mathcal{U}^{\mathcal{B}}{}_{\mathcal{D}} \varepsilon^{\mathcal{C}\mathcal{D}}, \quad (30)$$

and unitary

$$\varsigma_{\mathcal{A}\mathcal{B}} = \bar{\mathcal{U}}^{\mathcal{C}}{}_{\mathcal{A}} \mathcal{U}^{\mathcal{D}}{}_{\mathcal{B}} \varsigma_{\mathcal{C}\mathcal{D}} \quad (31)$$

(cf. Appendix). To prove unimodularity we use properties (7), (8), (9):

$$\begin{aligned}
\det \mathcal{U} &= \frac{1}{2} \mathcal{U}_{AB} \mathcal{U}^{AB} \\
&= \omega_A(\nu, p) \pi^A(S\nu, p) \bar{\omega}_{A'}(\nu, p) \bar{\pi}^{A'}(S\nu, p) + \omega_A(\nu, p) \omega^A(S\nu, p) \bar{\omega}_{A'}(\nu, p) \bar{\omega}^{A'}(S\nu, p) \\
&= \omega_A(\nu, p) \pi^A(S\nu, p) \omega_B(S\nu, p) \pi^B(\nu, p) + \omega_A(\nu, p) \omega^A(S\nu, p) \pi_B(\nu, p) \pi^B(S\nu, p) \\
&= \omega_A(\nu, p) \left[ \omega_B(S\nu, p) \pi^A(S\nu, p) - \pi_B(S\nu, p) \omega^A(S\nu, p) \right] \pi^B(\nu, p) = \omega_A(\nu, p) \pi^A(\nu, p) = 1. \quad (32)
\end{aligned}$$

Unitarity follows from unimodularity and

$$\mathcal{U}^0{}_0 = \overline{\mathcal{U}^1{}_1} = \bar{\mathcal{U}}^0{}_0, \quad (33)$$

$$\mathcal{U}^1{}_0 = -\overline{\mathcal{U}^0{}_1} = -\bar{\mathcal{U}}^1{}_0. \quad (34)$$

It is appropriate to recall the form of the scalar product the representation  $S \rightarrow U(S)$  is unitary with respect to. Let  $d\mu_m(p)$  denote an invariant measure on the mass- $m$  hyperboloid (cf. Eq. (25) in Part I). The scalar product is derived from the norm (130) in Part I which, using the  $\varsigma_{AB}$  BW-spinor, can be written as

$$\begin{aligned}
\| \psi_{\pm\alpha} \|^2 &= \int d\mu_m(p) \frac{\omega^A(\nu, p) \bar{\omega}^{A'}(\nu, p) \left( \psi_{\pm}(\pm p)_A^0 \overline{\psi_{\pm}(\pm p)_A^0} + \psi_{\pm}(\pm p)_A^1 \overline{\psi_{\pm}(\pm p)_A^1} \right)}{\omega^a(\nu, p) p_a} \\
&= \int d\mu_m(p) N^{-2} \omega^\alpha(\nu, p) \bar{\omega}^{\alpha'}(\nu, p) \psi_{\pm}(\pm p)_\alpha^A \bar{\psi}_{\pm}(\pm p)_{\alpha'}^B \varsigma_{AB} \\
&= \int d\mu_m(p) f_\pm^A(\pm p) \bar{f}_\pm^B(\pm p) \varsigma_{AB} = \int d\mu_m(p) \left( |f_\pm^0(\pm p)|^2 + |f_\pm^1(\pm p)|^2 \right). \quad (35)
\end{aligned}$$

The generalization to BW fields of arbitrary spin is immediate: The norm (56) from Part I becomes

$$\| \psi_{\pm\alpha_1 \dots \alpha_n} \|^2 = \int d\mu_m(p) f_\pm(\pm p)^{A_1 \dots A_n} \bar{f}_\pm(\pm p)^{B_1 \dots B_n} \varsigma_{A_1 B_1} \dots \varsigma_{A_n B_n}. \quad (36)$$

The simplicity of formulas has been achieved because of the simultaneous use of  $t^a = \omega^A(\nu, p) \bar{\omega}^{A'}(\nu, p)$  in the generalized norm  $\| \cdot \|'$ , and in the expansion of the Dirac bispinor in eigenstates of the P-L vector's projection in the *same*  $t^a$  direction. The simplicity is lost if non-null  $t^a$  are used since only null world-vectors factorize. The privileged role played by the BW amplitudes obtained with the help of the  $p$ -dependent spin-frames suggests that they deserve a name of their own to distinguish them from the standard “helicity” BW amplitudes. We will call the vectors

$$f^A = \begin{pmatrix} f^0 \\ f^1 \end{pmatrix} \quad (37)$$

the *BW-spinors*. BW-spinors can be also regarded as  $SU(2)$  spinor fields on a mass hyperboloid.

## V. PROOF OF $U(S')U(S) = U(S'S)$ ( $M \neq 0$ )

Passive transformations (27) of the BW-spinors form a unitary representation of the Poincaré group. The composition property  $U(S')U(S) = U(S'S)$  is proved by the following calculation:

$$\begin{aligned}
& \begin{pmatrix} U(S')[U(S)f]_{\pm}^0(\nu, \pm p) \\ U(S')[U(S)f]_{\pm}^1(\nu, \pm p) \end{pmatrix} \\
&= \begin{pmatrix} \omega_A(\nu, p)\pi^A(S'\nu, p), \mp\omega_A(\nu, p)\omega^A(S'\nu, p) \\ \pm\bar{\omega}_{A'}(\nu, p)\bar{\omega}^{A'}(S'\nu, p), \bar{\omega}_{A'}(\nu, p)\bar{\pi}^{A'}(S'\nu, p) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \omega_B(\nu, \Lambda'^{-1}p)\pi^B(S\nu, \Lambda'^{-1}p), \mp\omega_B(\nu, \Lambda'^{-1}p)\omega^B(S\nu, \Lambda'^{-1}p) \\ \pm\bar{\omega}_{B'}(\nu, \Lambda'^{-1}p)\bar{\omega}^{B'}(S\nu, \Lambda'^{-1}p), \bar{\omega}_{B'}(\nu, \Lambda'^{-1}p)\bar{\pi}^{B'}(S\nu, \Lambda'^{-1}p) \end{pmatrix} \begin{pmatrix} f_{\pm}^0(\nu, \pm(\Lambda'\Lambda)^{-1}p) \\ f_{\pm}^1(\nu, \pm(\Lambda'\Lambda)^{-1}p) \end{pmatrix} \\
&= \begin{pmatrix} \omega_A(\nu, p)\pi^A(S'\nu, p), \mp\omega_A(\nu, p)\omega^A(S'\nu, p) \\ \pm\bar{\omega}_{A'}(\nu, p)\bar{\omega}^{A'}(S'\nu, p), \bar{\omega}_{A'}(\nu, p)\bar{\pi}^{A'}(S'\nu, p) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \omega_B(S'\nu, p)\pi^B(S'S\nu, p), \mp\omega_B(S'\nu, p)\omega^B(S'S\nu, p) \\ \pm\bar{\omega}_{B'}(S'\nu, p)\bar{\omega}^{B'}(S'S\nu, p), \bar{\omega}_{B'}(S'\nu, p)\bar{\pi}^{B'}(S'S\nu, p) \end{pmatrix} \begin{pmatrix} f_{\pm}^0(\nu, \pm(\Lambda'\Lambda)^{-1}p) \\ f_{\pm}^1(\nu, \pm(\Lambda'\Lambda)^{-1}p) \end{pmatrix} \\
&= \begin{pmatrix} \omega_A(\nu, p)\pi^A(S'S\nu, p), \mp\omega_A(\nu, p)\omega^A(S'S\nu, p) \\ \pm\bar{\omega}_{A'}(\nu, p)\bar{\omega}^{A'}(S'S\nu, p), \bar{\omega}_{A'}(\nu, p)\bar{\pi}^{A'}(S'S\nu, p) \end{pmatrix} \begin{pmatrix} f_{\pm}^0(\nu, \pm(\Lambda'\Lambda)^{-1}p) \\ f_{\pm}^1(\nu, \pm(\Lambda'\Lambda)^{-1}p) \end{pmatrix} \\
&= \begin{pmatrix} U(S'S)f_{\pm}^0(\nu, \pm p) \\ U(S'S)f_{\pm}^1(\nu, \pm p) \end{pmatrix}.
\end{aligned}$$

We have used here the following two sequences of identities:

$$\begin{aligned}
& \omega_A(\nu, p)\pi^A(S'\nu, p)\omega_B(S'\nu, p)\pi^B(S'S\nu, p) - \omega_A(\nu, p)\omega^A(S'\nu, p)\bar{\omega}_{B'}(S'\nu, p)\bar{\omega}^{B'}(S'S\nu, p) \\
&= \omega_A(\nu, p)\pi^A(S'\nu, p)\omega_B(S'\nu, p)\pi^B(S'S\nu, p) - \omega_A(\nu, p)\omega^A(S'\nu, p)\pi_B(S'\nu, p)\pi^B(S'S\nu, p) \\
&= \omega_A(\nu, p)\left[\omega_B(S'\nu, p)\pi^A(S'\nu, p) - \pi_B(S'\nu, p)\omega^A(S'\nu, p)\right]\pi^B(S'S\nu, p) \\
&= \omega_A(\nu, p)\pi^A(S'S\nu, p),
\end{aligned} \tag{38}$$

and

$$\omega_A(\nu, p)\pi^A(S'\nu, p)\omega_B(S'\nu, p)\omega^B(S'S\nu, p) + \omega_A(\nu, p)\omega^A(S'\nu, p)\bar{\omega}_{B'}(S'\nu, p)\bar{\pi}^{B'}(S'S\nu, p)$$



$$\begin{aligned}
&= \omega_A(\nu, p) \pi^A(S' \nu, p) \omega_B(S' \nu, p) \omega^B(S' S \nu, p) + \omega_A(\nu, p) \omega^A(S' \nu, p) \omega_B(S' S \nu, p) \pi^B(S' \nu, p) \\
&= \omega_A(\nu, p) \omega_B(S' \nu, p) \pi^A(S' \nu, p) \omega^B(S' S \nu, p) - \omega_A(\nu, p) \pi_B(S' \nu, p) \omega^A(S' \nu, p) \omega^B(S' S \nu, p) \\
&= \omega_A(\nu, p) \left[ \omega_B(S' \nu, p) \pi^A(S' \nu, p) - \pi_B(S' \nu, p) \omega^A(S' \nu, p) \right] \omega^B(S' S \nu, p) \\
&= \omega_A(\nu, p) \omega^A(S' S \nu, p).
\end{aligned} \tag{39}$$

## VI. TRANSFORMATION PROPERTIES OF AMPLITUDES FOR $M = 0$

The Hertz-type form of solutions of the massless BW equations discussed in Part I leads to a single BW amplitude  $f_{\pm}(\pm p)$  for a massless field whose spin is arbitrary. This fact agrees with the general theorem stating that a massless finite-spin irreducible unitary representation of the Poincaré group must be induced by a one-dimensional representation.

An active transformation of the massless spinor field induces a passive transformation of the amplitude:

$$\begin{aligned}
\psi'_{\pm}(\pm p)_{A_1 \dots A_n}^{0 \dots 0} &= \pi_{A_1}(\nu, p) \dots \pi_{A_n}(\nu, p) f'_{\pm}(\nu, n, \pm p)^{0 \dots 0} \\
&= S_{A_1}^{B_1} \dots S_{A_n}^{B_n} \pi_{B_1}(\nu, \Lambda^{-1} p) \dots \pi_{B_n}(\nu, \Lambda^{-1} p) f_{\pm}(\nu, n, \pm \Lambda^{-1} p)^{0 \dots 0}.
\end{aligned} \tag{40}$$

The passive transformation of the amplitude is (compare (18))

$$\begin{aligned}
U(S) f_{\pm}(\nu, n, \pm p)^{0 \dots 0} &= \left[ \omega^A(\nu, n, p) \pi_A(S \nu, p) \right]^n f_{\pm}(\nu, n, \pm \Lambda^{-1} p)^{0 \dots 0} \\
&= \left[ \frac{p^{AA'} \nu_A \bar{S}_{A'}^{B'} \bar{\nu}_{B'}}{[p^{CC'} \nu_C \bar{S}_{C'}^{D'} \bar{\nu}_{D'}]} \right]^n f_{\pm}(\nu, n, \pm \Lambda^{-1} p)^{0 \dots 0}
\end{aligned} \tag{41}$$

$$= \mathcal{U}(S, \nu, p) f_{\pm}(\nu, n, \pm \Lambda^{-1} p)^{0 \dots 0}. \tag{42}$$

(41) shows that  $\mathcal{U}(S, \nu, p)$  is a phase factor and hence the transformation is unitary.

Had we started with a massless field having  $n$  primed indices

$$\psi_{\pm}(\pm p)_{A'_1 \dots A'_n}^{I \dots I} = \bar{\pi}_{A'_1}(\nu, p) \dots \bar{\pi}_{A'_n}(\nu, p) f_{\pm}(\nu, n, \pm p)^{I \dots I}, \tag{43}$$

we would have obtained a complex-conjugated transformation rule

$$U(S) f_{\pm}(\nu, n, \pm p)^{I \dots I} = \bar{\mathcal{U}}(S, \nu, p) f_{\pm}(\nu, n, \pm \Lambda^{-1} p)^{I \dots I}. \tag{44}$$

It is interesting that the sign-of-energy index “ $\pm$ ” is not necessary in either  $\mathcal{U}(S, \nu, p)$  or  $\bar{\mathcal{U}}(S, \nu, p)$ . In the  $m \neq 0$  case these signs entered the transformation properties *via* the off-diagonal elements of the  $SU(2)$  matrices. Here the off-diagonal elements do not appear since the representation is one-dimensional.

## VII. PROOF OF $U(S')U(S) = U(S'S)$ ( $M = 0$ )

It is sufficient to prove the composition property  $U(S')U(S) = U(S'S)$  for a “ $\theta \dots \theta$ ” amplitude:

$$\begin{aligned}
U(S')[U(S)f]_{\pm}(\nu, n, \pm p)^{\theta \dots \theta} &= \mathcal{U}(S', \nu, p)\mathcal{U}(S, \nu, \Lambda'^{-1}p)f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} \\
&= \left[ \omega^A(\nu, n, p)\pi_A(S'\nu, p)\omega^B(\nu, n, \Lambda'^{-1}p)\pi_B(S\nu, \Lambda'^{-1}p) \right]^n f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} \\
&= \left[ \omega^A(\nu, n, p)\pi_A(S'\nu, p)\omega^B(S'\nu, S'n, p)\pi_B(S'S\nu, p) \right]^n f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} \\
&= \left[ \omega^A(\nu, n, p)\left\{ \varepsilon_A^B + \omega_A(S'\nu, S'n, p)\pi^B(S'\nu, p) \right\} \pi_B(S'S\nu, p) \right]^n f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} \\
&= \left[ \omega^A(\nu, n, p)\pi_A(S'S\nu, p) \right]^n f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} \\
&= \mathcal{U}(S'S, \nu, p)f_{\pm}(\nu, n, \pm(\Lambda'\Lambda)^{-1}p)^{\theta \dots \theta} = U(S'S)f_{\pm}(\nu, n, \pm p)^{\theta \dots \theta}. \tag{45}
\end{aligned}$$

We have used here the fact that  $\pi_B(S'S\nu, p)$  and  $\pi_B(S'\nu, p)$  are proportional (see (18)) — technically this property of  $\pi$ -spinors is responsible for the one-dimensionality of the representation and is typical only of null momenta. An analog of (36) can be introduced also for the massless fields. As an example consider again a field having  $n$ -unprimed spinor indices. The corresponding index type of the BW-spinor amplitude is represented by  $n$   $\theta$ 's. Let us first trivially “embed” the amplitude in the BW-spinor:

$$f_{\pm}(\nu, n, \pm p)^{A_1 \dots A_n} = \begin{pmatrix} f_{\pm}(\nu, n, \pm p)^{\theta \dots \theta} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{46}$$

Now

$$\begin{aligned}
\| \psi_{\pm\alpha_1\ldots\alpha_n} \|^2 &= \int d\mu_0(p) f_{\pm}(\pm p)^{A_1\ldots A_n} \bar{f}_{\pm}(\pm p)^{B_1\ldots B_n} \zeta_{A_1 B_1} \cdots \zeta_{A_n B_n} \\
&= \int d\mu_0(p) |f_{\pm}(\nu, n, \pm p)^{0\ldots 0}|^2.
\end{aligned} \tag{47}$$

The embedding (46) unifies the massive and massless cases because the spinor (46) is a true  $SU(2)$  BW-spinor as opposed to the amplitude  $f_{\pm}(\nu, n, \pm p)^{0\ldots 0}$  which, taken alone, should be regarded as a  $U(1)$  field.

### VIII. SUMMARY AND DISCUSSION

The well known BW scalar product is a particular case of a larger class of scalar products parametrized by a family of world-vectors. If the world-vectors are null and  $p$ -dependent then the BW amplitudes play a role of momentum-space wave functions corresponding to projections of the Pauli-Lubanski vector in these momentum-dependent null directions. The choice of null directions leads to a simplification of the formalism because of the factorization property of null world-vectors. The BW amplitudes constructed in this way transform as scalar fields under the action of active  $SL(2, C)$  transformations. The corresponding passive transformations of the amplitudes are local (i.e.  $p$ -dependent)  $SU(2)$ . The BW indices, which originally played a role of binary numbering of different irreducible components of the spinor BW field, turn out to play a dual role of BW-spinor indices if the passive transformations are concerned. This property leads to a BW-analog of the ordinary spinor algebra.

There exist also other interesting formal analogies between the BW-spinor and 2-spinor formalisms. For example, the BW-spinors are obtained as contractions of 2-spinor indices of BW fields with  $\omega$ -spinors. This is analogous to the way Penrose *et al.* introduce spin-weighted spherical harmonics [3] but here everything happens in the momentum-space. It seems that the similarities between the two approaches are worth of further studies. This includes the question of the role of conformal symmetries (typical of *null* formalisms) and relations to twistors.

The fact that pairs of null directions corresponding to spin-frames simplify the description of infinite-dimensional unitary representations of the Poincaré group goes hand-in-hand with a general

philosophy underlying the spinor approach to field theories and space-time geometry. In a forthcoming paper we shall discuss implications of the generalized formalism for the structure of generators. One may expect that the null formalism will be related to Dirac's front form of generators [5].

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## X. APPENDIX: BISPINORS VS. BW-SPINORS

An unprimed lower-index bispinor is

$$\psi_\alpha = \begin{pmatrix} \psi_A^0 \\ \psi_{A'}^1 \end{pmatrix}. \quad (48)$$

Complex conjugated bispinors are

$$\bar{\psi}_{\alpha'} = \overline{\psi_\alpha} = \begin{pmatrix} \overline{\psi_A^0} \\ \overline{\psi_{A'}^1} \end{pmatrix} = \begin{pmatrix} \bar{\psi}_{A'}^1 \\ \bar{\psi}_A^0 \end{pmatrix}. \quad (49)$$

Let

$$\psi_\alpha^0 = \begin{pmatrix} \psi_A^0 \\ 0 \end{pmatrix}, \quad \psi_\alpha^1 = \begin{pmatrix} 0 \\ \psi_{A'}^1 \end{pmatrix}, \quad \bar{\psi}_{\alpha'}^1 = \begin{pmatrix} \bar{\psi}_{A'}^1 \\ 0 \end{pmatrix}, \quad \bar{\psi}_{\alpha'}^0 = \begin{pmatrix} 0 \\ \bar{\psi}_A^0 \end{pmatrix}, \quad (50)$$

$$\omega_\alpha = \begin{pmatrix} \omega_A \\ \bar{\omega}_{A'} \end{pmatrix}, \quad \bar{\omega}_{\alpha'} = \begin{pmatrix} \bar{\omega}_{A'} \\ \omega_A \end{pmatrix}. \quad (51)$$

Then

$$\omega^\alpha \psi_\alpha^0 = \omega^A \psi_A^0 = \psi^0, \quad (52)$$

$$\omega^\alpha \psi_\alpha^1 = \bar{\omega}^{A'} \psi_{A'}^1 = \psi^1, \quad (53)$$

$$\bar{\omega}^{\alpha'} \bar{\psi}_{\alpha'}^1 = \bar{\omega}^{A'} \bar{\psi}_{A'}^1 = \bar{\psi}^1, \quad (54)$$

$$\bar{\omega}^{\alpha'} \bar{\psi}_{\alpha'}^0 = \omega^A \psi_A^0 = \bar{\psi}^0. \quad (55)$$

The following expression appears often in connection with BW scalar products:

$$\begin{aligned}\omega^A \bar{\omega}^{A'} \left( \psi_A^0 \bar{\psi}_A^0 + \psi_{A'}^1 \bar{\psi}_{A'}^1 \right) &= \omega^A \bar{\omega}^{A'} \left( \psi_A^0 \bar{\psi}_{A'}^1 + \psi_{A'}^1 \bar{\psi}_A^0 \right) = \omega^\alpha \bar{\omega}^{\alpha'} \left( \psi_\alpha^0 \bar{\psi}_{\alpha'}^1 + \psi_{\alpha'}^1 \bar{\psi}_\alpha^0 \right) \\ &= \omega^\alpha \bar{\omega}^{\alpha'} \psi_\alpha^A \bar{\psi}_{\alpha'}^B \varsigma_{AB} = \psi^A \bar{\psi}^B \varsigma_{AB},\end{aligned}\tag{56}$$

where

$$\varsigma_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\varsigma^{AB}.\tag{57}$$

To express covariantly unimodularity of the BW transformation matrices we introduce the BW-spinor version of  $\varepsilon$ -spinors:

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{AB}.\tag{58}$$

(58) are used to raise or lower the BW-spinor indices.

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